18.152 PROBLEM SET 4 SOLUTIONS

DONGHAO WANG

1. Problem 4

The last problem of this p-set is extremely challenging. Only one student figured this problem out. The solution below is based upon the work of Dhruv Rohatgi.

Many students realized the relation of Problem 4 with the fundamental Schauder estimate in Lecture 2:

Theorem 1.1. *For any $0 < \alpha < 1$ and $n \geqslant 1$, there exists a constant $C(n, \alpha) > 0$ such that the inequality*

$$[D^2 u]_\alpha \leqslant C[\Delta u]_\alpha$$

holds for any function $u \in C^{2,\alpha}(\mathbb{R}^n)$.

The Schwarz reflection principle in Problem 3 only applies to a harmonic function, but in Problem 4, you start off with a general function

$$u \in C^{2,\alpha}(\bar{\mathbb{R}}^n_+).$$

In fact, you are supposed to modify the proof of Theorem 1.1 and try to apply the Schwarz reflection principle at some intermediate step.

Recall first that the semi-norm $[\cdot]_{\alpha,\mathbb{R}^n}$ is defined as

$$[u]_{\alpha,\mathbb{R}^n} = \sup_{x,y\in\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

We introduce a more convenient semi-norm that is actually equivalent to $[u]_{\alpha,\mathbb{R}^n}$:

Lemma 1.2. *For any $u \in C^{2,\alpha}(\mathbb{R}^n)$, define*

$$[u]'_\alpha := \sup_{x\in\mathbb{R}^n, h>0, 1\leqslant k\leqslant n} \frac{|u(x + he_k) - u(x)|}{h^\alpha},$$

where $e_k = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the basic vector in $\mathbb{R}^n$ with k-th entry equal to 1 and all remaining entries vanishing. Then for some dimensional constant $C_1(n) > 0$,

$$[u]'_\alpha \leqslant [u]_\alpha \leqslant C_1 \cdot [u]'_\alpha.$$

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Proof. It is clear that $[u]'_\alpha \leqslant [u]_\alpha$. For the second inequality, take $x, y \in \mathbb{R}^n$ and define

$$x_0 = x, x_1, \cdots, x_n = y$$

such that

$$x_k - x_{k-1} = a_k \cdot e_k,\ a_k \in \mathbb{R}.$$

We compute:

$$|u(x) - u(y)| \leqslant \sum_{k=1}^{n} |u(x_k) - u(x_{k-1})| \leqslant \sum_{k=1}^{n} |a_k|^\alpha [u]'_\alpha \leqslant n \cdot |x-y|^\alpha [u]'_\alpha,$$

or

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \leqslant n[u]'_\alpha.$$

Since $x, y \in \mathbb{R}^n$ are arbitrary, $[u]_\alpha \leqslant n[u]'_\alpha$. □

Lemma 1.2 also applies to $\bar{\mathbb{R}}^n_+$. Now we are ready to prove Problem 4.

Solution to Problem 4. Suppose on the contrary that there exists a sequence of function $\{u_m\} \subset C^{2,\alpha}(\bar{\mathbb{R}}^n_+)$ such that

$$[D^2 u_m]_\alpha \geqslant m[\Delta u_m]_\alpha.$$

Let

$$v_m = \frac{u_m}{[D^2 u_m]'_\alpha},$$

then $[D^2 v_m]'_\alpha = 1$ and by Lemma 1.2

$$[\Delta v_m]_\alpha = \frac{[\Delta u_m]_\alpha}{[D^2 u_m]'_\alpha} \leqslant \frac{[D^2 u_m]_\alpha}{m[D^2 u_m]'_\alpha} \leqslant \frac{C_1}{m} \to 0.$$

In particular, we can find some $x_m \in \bar{\mathbb{R}}^n_+$, $h_m > 0$ and $k_m, i_m, j_m \in \{1, \cdots, n\}$ such that

$$\frac{|\partial_{i_m}\partial_{j_m} v_m(x_m + h_m e_{k_m}) - \partial_{i_m}\partial_{j_m} v_m(x_m)|}{h_m^\alpha} > \frac{1}{2}.$$

Since k_m, i_m, j_m can only take finite possible values, by passing to a subsequence of $\{u_m\}$, we may assume $k_m = k, i_m = i, j_m = j \in \{1, \cdots, n\}$ are independent of the subscript m; so

$$\frac{|\partial_i\partial_j v_m(x_m + h_m e_k) - \partial_i\partial_j v_m(x_m)|}{h_m^\alpha} > \frac{1}{2}. \tag{1}$$

Let x_m^n be the last coordinate of x_m. The most difficult part of Problem 4 is to realize that we have to address two different cases in different ways.

- $\limsup_{m\to\infty} x_m^n / h_m = \infty$;

- $\limsup_{m\to\infty} x_m^n/h_m < \infty$;

We start with the first case, which is slightly simpler.

Case 1. By passing to a subsequence, assume

$$\lim_{m\to\infty} \frac{x_m^n}{h_m} = \infty; \tag{2}$$

Apply the rescaling argument to the pair (x_m, h_m) and define

$$w_m(x) = \frac{v_m(h_m x + x_m)}{h_m^{2+\alpha}}.$$

Then the function w_m is defined on the sub-domain

$$\{x \in \mathbb{R}^n : x_n \geqslant -\frac{x_m^n}{h_m}\}.$$

By (1)

$$|\partial_i\partial_j w_m(e_k) - \partial_i\partial_j w_m(0)| > \frac{1}{2}.$$

and $[D^2 w_m]_\alpha = 1$. Finally, set

$$p_m(x) = w_m(x) - w_m(0) - x^T\nabla w_m(0) - \frac{1}{2}x^T\nabla^2 w_m(0)x.$$

Now we can proceed as in the proof of Theorem 1.1 and use the Arzela-Ascoli lemma to show

$$p_m \to p_\infty \in C^{2,\alpha}(\mathbb{R}^n)$$

uniformly on the ball $B_R(0)$ for any $R > 0$. The point is that the limit p_∞ is defined on the whole space $\mathbb{R}^n$ because of the condition (2). Since p_∞ is also harmonic, the proof of Theorem 1.1 can now proceed with no difficulty. Details are omitted here.

Case 2. Now we address the second case. By passing to a subsequence of $\{v_m\}$, assume the limit

$$\lim_{m\to\infty} \frac{x_m^n}{h_m} = T < \infty \tag{3}$$

is finite; so when $m \gg 1$,

$$x_m^n \leqslant (1 + T) \cdot h_m. \tag{4}$$

The idea is **to change the point** x_m **into** y_m such that

- the last coordinate $y_m^n = 0$;

- the inequality (1) holds for y_m with a possibly smaller constant, i.e

$$\frac{|\partial_i\partial_j v_m(y_m + s_m e_n) - \partial_i\partial_j v_m(y_m)|}{s_m^\alpha} > \epsilon, \tag{5}$$

for some $\epsilon > 0$, $s_m > 0$.

The rescaling argument will be applied to the pair (y_m, s_m). To do this, write $x_m = (x'_m, x^n_m)$ and set

$$z_m = (x'_m, 0),$$

i.e. z_m is the projection of x_m on the boundary $\mathbb{R}^{n-1} \times \{0\} \subset \bar{\mathbb{R}}^n_+$. If $k \neq n$, consider the four points:

$$x_m, z_m, z_m + h_m e_k, x_m + h_m e_k.$$

Lemma 1.3. *At least one of the following inequalities holds:*

$$\frac{|\partial_i\partial_j v_m(x_m) - \partial_i\partial_j v_m(z_m)|}{|x^n_m|^\alpha} \geqslant \frac{1}{4(1+T)^\alpha},$$
$$\frac{|\partial_i\partial_j v_m(z_m + h_m e_k) - \partial_i\partial_j v_m(x_m + h_m e_k)|}{|x^n_m|^\alpha} \geqslant \frac{1}{4(1+T)^\alpha}.$$

Proof of Lemma. Otherwise, we have

$$|\partial_i\partial_j v_m(x_m) - \partial_i\partial_j v_m(z_m)| \leqslant \frac{|x^n_m|^\alpha}{4(1+T)^\alpha} \leqslant \frac{h_m^\alpha}{4},$$
$$|\partial_i\partial_j v_m(z_m + h_m e_k) - \partial_i\partial_j v_m(x_m + h_m e_k)| \leqslant \frac{|x^n_m|^\alpha}{4(1+T)^\alpha} \leqslant \frac{h_m^\alpha}{4},$$

by (4). Note also $\partial_i\partial_j v_m(z_m + h_m e_k) = \partial_i\partial_j v_m(z_m) = 0$, since v_m is identically zero on the boundary $\mathbb{R}^{n-1} \times \{0\}$. By adding them together, we reach a contradiction with the inequality (1). □

Return to the solution of Problem 4. If $k \neq n$, then we make take $y_m = z_m$ or $z_m + h_m e_k$ depending on which inequality holds in Lemma 1.3. Take $s_m = x^n_m$ in (5).

If $k = n$, we take $y_m = z_m$, the projection of x_m on the boundary. The length s_m is either x^n_m or $x^n_m + h_m$. Finally, take ϵ as

$$\frac{1}{4(T+2)^\alpha}.$$

Now we apply the rescaling argument with respect to (y_m, s_m). Define

$$w_m(x) = \frac{v_m(s_m x + y_m)}{s_m^{2+\alpha}}.$$

Then the function w_m is still defined on the space

$$\bar{\mathbb{R}}^n_+ := \{x \in \mathbb{R}^n : x_n \geqslant 0\}$$

and by (5)

$$|\partial_i \partial_j w_m(e_k) - \partial_i \partial_j w_m(0)| > \frac{1}{2}.$$

Moreover,

$$\begin{aligned} [D^2 w_m]_\alpha &= 1, \\ [\Delta w_m]_\alpha &\leqslant 1/n, \\ w_m &\equiv 0 \text{ on } \mathbb{R}^{n-1} \times \{0\}. \end{aligned}$$

Finally, set

$$p_m(x) = w_m(x) - w_m(0) - x^T \nabla w_m(0) - \frac{1}{2} x^T \nabla^2 w_m(0) x.$$

Then

$$\begin{aligned} [D^2 p_m]_\alpha &= 1, \\ [\Delta p_m]_\alpha &\leqslant 1/n, \\ p_m &\equiv 0 \text{ on } \mathbb{R}^{n-1} \times \{0\}, \\ p_m(0) &= 0,\ \nabla p_m(0) = 0,\ \nabla^2 p_m(0) = 0, \\ p_m(x) &\equiv 0 \text{ on } \mathbb{R}^{n-1} \times \{0\}. \end{aligned}$$

Now we can proceed as in the proof of Theorem 1.1 and use the Arzela-Ascoli lemma to show

$$p_m \to p_\infty \in C^{2,\alpha}(\bar{\mathbb{R}}^n_+)$$

uniformly on the ball $B_R(0) \cap \bar{\mathbb{R}}^n_+$ for any $R > 0$ (the convergence is not in $C^{2,\alpha}$ in general, but the limit p_∞ does lie in this space). The limit p_∞ is defined only on $\bar{\mathbb{R}}^n_+$. Moreover,

$$\begin{aligned} [D^2 p_\infty]_\alpha &\leqslant 1, \\ [\Delta p_\infty]_\alpha &= 0, \\ p_\infty &\equiv 0 \text{ on } \mathbb{R}^{n-1} \times \{0\}, \\ p_\infty(0) &= 0,\ \nabla p_\infty(0) = 0,\ \nabla^2 p_\infty(0) = 0, \\ p_\infty(x) &\equiv 0 \text{ on } \mathbb{R}^{n-1} \times \{0\}. \end{aligned}$$

In particular, p_∞ is a harmonic function by the second and the fourth properties. Now we apply the Schwartz reflection principle to extend p_∞ to the whole space $\mathbb{R}^n$. The rest of the proof follows the same line of arguments as in Theorem 1.1. Details are omitted here. □